RESONANT SPEEDS AND VIBRATIONAL RESPONSE OF A PRESTRESSED RECTANGULAR PLATE TRAVELLED BY MOVING CONCENTRATED LOAD

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ABSTRACT

An investigation of the effects of axial force and foundation stiffness on the oscillatory motions of a simply supported rectangular plate excited by moving concentrated load is presented. The resonant states and the influence of these physical parameters on the associated critical speeds of motion of the dynamical system are also examined. In order to obtain the approximate analytical solution of the dynamical plate problem the method of integral transformation is used, in particular the finite Fourier Sine transform which reduces the fourth-order partial differential equation of motion to a second-order ordinary differential equation on one hand while on the other, the resulting governing problem is then solved analytically in the time domain using the Laplace integral transformation method to obtain the expression for the transverse deflection of the rectangular plate structure under harmonic moving load. Numerical illustration of this solution is presented in plotted curves. From the displayed graphs, one observes that increases in the axial force values for a fixed value of foundation stiffness results in a decrease in the amplitude of deflection. Thus, prestressed structures experienced reduced deflection and an enhanced load carrying capacity. Also, increases in the values of foundation stiffness for a fixed value of the axial force give rise to a reduction in the amplitude of deflection of the plate.Besides; the condition under which the dynamical system may experience resonance phenomena and the respective resonant speeds (critical speeds) were obtained and shown in plotted curves. The graphs reveal that the critical speed of the dynamical system decreases with increase in the values of axial force. The effect of axial force on the critical speed of motion is more pronounced for very high values of axial force suggesting that the structural designs are more stable and reliable for higher values of axial force. In the same vein, the critical speed decreases with an increase in the effects of foundation stiffness, thus evading the tendency of excessive deflections and so limiting the catastrophe of resonance.

Keywords: Axial force, foundation stiffness, critical speed, resonance, rectangular plate.

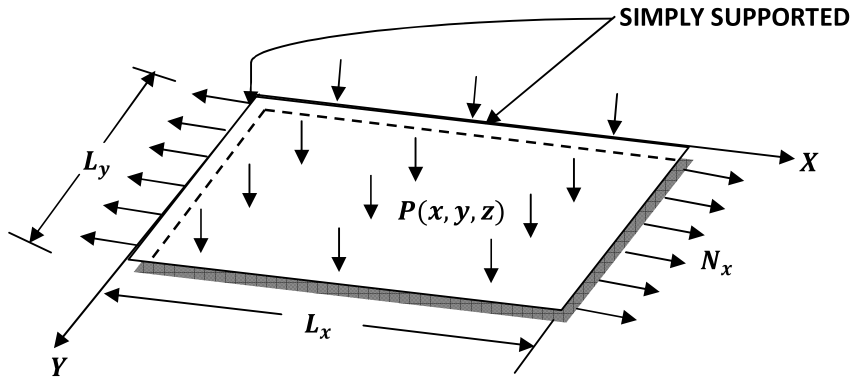
1.0 INTRODUCTION

Extensive research has been carried out by authors in the fields of engineering, physics and applied mathematics with regard to the vibration analysis of plate or plate-type structures traversed by moving concentrated loads owing to its relevance in many diverse areas. Plates have gained special importance and notably increased applications in recent years. A large number of structural components in engineering structures can be classified as plates. Typical examples in civil engineering structures are floor and foundation slabs, lock-gates, thin retaining walls, bridge decks and slab bridges. Plates are also indispensable in shipbuilding and aerospace industries. Plates are also frequently parts of machineries and other mechanical devices. Foremost amongst researchers of a moving load on a plate is the work of Willis(1849) who investigated the effects of weights travelling over bars with different velocities. Others are Stokes (1849), Timoshenko (1922), Lowan(1935), Bondar(1954), Reissmann (1963) and the monograph of Fryba (1972) to mention but few. The operation of these moving loads – cars, heavy-duty vehicles, trucks, railways, steam and gas engines, etc – introduce additional dynamic stresses on the plate structure. The stresses induced in the plate are dependent not only on the magnitude of the loads, but also strongly upon their speed of propagation. From literatures Schmidt (1931) and Reissner (1932) have examined this phenomenon for simply-supported rectangular plates. Also Holl(1950) and Livesley (1953)considered the case of an infinite plate resting on elastic foundation and travelled by moving load. In, Schmidt (1931), Reissner (1932) and Holl (1950), critical speeds of propagation are shown to exist. When critical speed exists and the effect of damping is neglected, the deflections become unbounded when the load propagates with a speed equal to a critical speed. In the stage of design the effects of some physical phenomena like axial load, foundation stiffness on the oscillatory motions of these structures when traversed by loads are taken into consideration in order to improve their performance, extend their life span and also improve the riding quality of the moving load. Also worthy of mention are the works of Jahanshahi et al (1965) who studied the the effects of rotatory inertia and transverse shear on the response of elastic plate to moving forces by means of Mindlin's (1951) two-dimensional theory. Much later, Oni et al (2005) assessed the rotatory inertial influence on the highly prestressed orthotropic rectangular plate under the action of moving loads. They employed the method of composite expansion (MCE) in conjunction with the method of integral transformation and Cauchy residue theorem to obtain an approximate uniformly valid solution in the entire domain of definition of the rectangular plate. Recently, Tolorunshagba (2014) analysed the impact response of a non-uniform thin beam, resting on exponentially decaying foundation, to the passage of moving loads. He used the versatile Galerk in method and integral transformation to obtain the expression for the dynamic response of the plate under the action of moving loads.

Despite the relevance and versatility of the aforementioned related literatures, the researchers did not treat the effects of axial load and foundation on the dynamic response of plate or plate-like structures under the action of moving loads. However, this paper addresses the influence of axial load and elastic foundation (Winkler type) on the impact response of plates under the passage of moving loads and also takes into account the much dreaded resonance phenomenon and the resonant speeds at which the dynamical system experiences resonance.

2.0 BASIC FORMULATION

Consider a rectangular thin plate resting on an elastic foundation which is simultaneously loaded with uniform moving concentrated load P(x, y, t) and subjected to uniform edge-tension N_x at the edges $x = 0$ and $x = L_x$ as shown in the figure below.



Assume the material and the supporting medium of the initially flat plate under consideration is linearly elastic and its middle plane remain neutral during bending. Neglecting the effects of damping and shear deformation, the transverse motions of the rectangular thin plate, resting on constant elastic foundation, under the actions of uniform edge-tension and moving concentrated load is governed by the fourth-order partial differential equation of the form

$$D\left[\frac{\partial^4 v(x,y,t)}{\partial x^4} + 2\frac{\partial^4 v(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 v(x,y,t)}{\partial y^4}\right] + \mu\frac{\partial^2 v(x,y,t)}{\partial t^2} + Fv(x,y,t) = P(x,y,t) + N_x\frac{\partial^2 v(x,y,t)}{\partial x^2} + N_y\frac{\partial^2 v(x,y,t)}{\partial x^2} \quad (1)$$

Here D = Bending rigidity of the plate, μ = mass of the plate per unit area, $v(x, y, t)$ = transverse deflection of the plate, N_x=axial force in the x-direction, N_y= axial force in the y-direction, x = position coordinate in x-direction, y = position coordinate in y-direction, F =constant foundation stiffness, $P(x, y, t)$ = moving concentrated load.

Since the plate is simply supported, the pertinent boundary conditions are

$$V(x,y,t)|_{x=0,L_x} = 0 = \frac{\partial^2 v(x,y,t)}{\partial x^2}|_{x=0,L_x} \quad (2)$$

$$V(x,y,t)|_{y=0,L_y} = 0 = \frac{\partial^2 v(x,y,t)}{\partial x^2}|_{y=0,L_y} \quad (3)$$

The plate is also supposed to be at rest prior to the arrival of the moving load and so the initial conditions are taken to be

$$V(x,y,t)|_{t=0,} = 0 = \frac{\partial v(x,y,t)}{\partial t^2}|_{t=0,} \quad (4)$$

Assuming the moving load is harmonic and moves on the rectangular plate with a constant velocity c along a straight line parallel to the x-axis, say $y = y_o$, then the load P(x, y, t) takes the form

$$P(x, y, t) = P_o cos\lambda t\delta(x - ct)\delta(y - y_o) \quad (5)$$

whereP_o is the amplitude and λ is the driving circular frequency of the moving force, and $\delta(\cdot)$ is the generalized function known as Dirac delta function defined as

$$\delta(x-ct)=\begin{cases}0, & x\neq ct\\ \infty, & x=ct\end{cases} \qquad (6)$$

with the fundamental property

$$\int_a^b \delta(x-\xi t)\,f(x)dx = \begin{cases}0, & \xi\leq a\\ f(\xi), & a<\xi<b\\ 0, & \xi>b\end{cases} \qquad (7)$$

Suppose the rectangular plate is subjected to tensile stresses at the edges x=0 and $x=L_x$ only, then the axial force in the x- and y-directions take the formillustrated in Rudolph (2004):

$$N_x = N_0, \qquad N_y = 0 \qquad (8)$$

whereN_0 is the constant edge tension

Substituting equations (5) and (8)into equation (1) and rearrangingproduces

$$D\left[\frac{\partial^4 v(x,y,t)}{\partial x^4}+2\frac{\partial^4 v(x,y,t)}{\partial x^2\partial y^2}+\frac{\partial^4 v(x,y,t)}{\partial y^4}\right]+\mu\frac{\partial^2 v(x,y,t)}{\partial t^2}-N_0\frac{\partial^2 v(x,y,t)}{\partial x^2}+Fv(x,y,t)= p_o cos\lambda t\delta(x-ct)\delta(y-y_o) \qquad (9)$$

2.2 OPERATIONAL SIMPLIFICATION

The governing equation (9) is a fourth-order partial differential equation with singularity and variable coefficients.

In order to solve the differential equation (9), use is made of the double finite Fourier sine transform defined by

$$V(j,k,t)=\int_0^{ly}\int_0^{lx} v(x,y,t)\,sin\frac{j\pi x}{l_x}sin\frac{k\pi y}{l_y}dx\,dy$$

(10)

with inverse

$$v(x,y,t)=\frac{4}{l_x l_y}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mathrm{V}\,(\mathrm{j,k,t})\,sin\frac{j\pi x}{l_x}sin\frac{k\pi y}{l_y}dx$$

(11)

In view of equations (10), (2) and (3) and the property of Dirac delta function in equation (7), equation (9) becomes

$$V_{tt}(j,k,t)+\left\{\frac{D}{\mu}\left[\left(\frac{j\pi}{l_x}\right)^4+2\left(\frac{j\pi}{l_x}\right)^2\left(\frac{k\pi}{l_y}\right)^2+\left(\frac{k\pi}{l_y}\right)^4\right]-\frac{N_o}{\mu}\left(\frac{j\pi}{l_x}\right)^2+\frac{F}{\mu}\right\}V(j,k,t)= \frac{P_0}{\mu}cos\lambda t sin\frac{j\pi ct}{l_x}sin\frac{k\pi y_0}{l_y}$$

(12)

subject to the transformed initial conditions

$$V(j,k,t)|_{t=0}=0=V_t(j,k,t)|_{t=0}$$

(13)

Now equation (12) can be rewritten as

$$V_{tt}(j,k,t)+\alpha^2 V(j,k,t)=Pcos\lambda t sin\beta_j t$$

(14)

Where the subscript denote the derivative with respect to t and

$$\alpha^2 = \omega_{ij}^2 + H_1 + H_2, H_1 = \frac{N_o}{\mu}\left(\frac{j\pi}{l_x}\right)^2, \; H_2 = \frac{F}{\mu}, P = \frac{p_o}{\mu} sin\frac{k\pi y_o}{l_y}, \beta_j = \left(\frac{j\pi c}{l_x}\right)$$

(15)

and finally

$$\omega_{ij}^2 = \frac{D}{\mu}\left[\left(\frac{j\pi}{l_x}\right)^4 + 2\left(\frac{j\pi}{l_x}\right)^2\left(\frac{k\pi}{l_y}\right)^2 + \left(\frac{k\pi}{l_y}\right)^4\right]$$

(16)

is the natural circular frequency of the plate in free transverse vibration.

2.3 SOLUTION OF TRANSFORMED EQUATION

Evidently, equation (14) indicates that the boundary-initial value problem specified completely in equations (9), (2) – (4) has reduced to a second order linear ordinary differential equation with constant coefficients. This suggests an integral transformation treatment, in particular, the Laplace transformation defined by

$$V(j,k,s) = \int_0^\infty V(j,k,t)e^{-st}\,dt, \qquad s > 0$$

(17)

where s is a complex parameter

An application of equation (17) to equation (14) in conjunction with the transformed initial conditions in equation (13) yields the algebraic equation given by

$$V(j,k,s) = \frac{P}{2\alpha}\left[\frac{\alpha}{s^2+\alpha^2}\cdot\frac{\gamma_1}{s^2+\gamma_1^2} - \frac{\alpha}{s^2+\alpha^2}\cdot\frac{\gamma_2}{s^2+\gamma_2^2}\right]$$

(18)

in which

$$\gamma_1 = (\lambda + \beta_j), \qquad \gamma_2 = (\lambda - \beta_j)$$

(19)

In order to obtain the Laplace inversion of equation (18), use is made of the following representations.

$$G(s) = \frac{\alpha}{s^2+\alpha^2}, W_1(s) = \frac{\gamma_1}{s^2+\gamma_1^2}, \quad W_2(s) = \frac{\gamma_2}{s^2+\gamma_2^2}$$

(20)

with

$$g(t) = sin\alpha t, \quad w_1(t) = sin\gamma_1 t, w_2(t) = sin\gamma_2 t \qquad (21)$$

in which $G(s)$, $W_1(s)$, $W_2(s)$ are the respective Laplace transformation of $g(t),\ w_1(t), w_2(t)$.

Perform an inverse Laplace transformation of (18) using convolution integral defined by

$$V_i(j,k,t) = \int_0^t g(u)w_i(t-u)du, \quad i = 1,2.$$

(22)

to obtain

$$V(j,k,t) = \frac{P}{2\alpha}\left[\frac{1}{\alpha^2-\gamma_1^2}(\alpha sin\gamma_1 t - \gamma_1 sin\alpha t) - \frac{1}{\alpha^2-\gamma_2^2}(\alpha sin\gamma_2 t - \gamma_2 sin\alpha t)\right]$$

(23)

Consequently, equation (23) is substituted into equation(11) to obtain the inversion of the finite Fourier sine transformation of the boundary-initial value problem specified completely in equations (9), (2) – (4) as

$V(x,y,t) =$
$\frac{2}{l_x l_y}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{P}{\alpha}\left[\frac{1}{\alpha^2-\gamma_1^2}(\alpha sin\gamma_1 t - \gamma_1 sin\alpha t) - \frac{1}{\alpha^2-\gamma_2^2}(\alpha sin\gamma_2 t -\right.$
$\gamma 2 sin\alpha t sin j\pi x lx sin k\pi y ly$(24)

Equation (24) is the displacement response of the simply supported thin plate subjected to simultaneous bending and stretching forces.

3.0 NUMERICAL SIMULATIONSAND DISCUSSION OF ANALYTICAL RESULT

This section presents the analysis of analytical results and numerical illustrations related to the solution of the reinforced thin rectangular plate carrying a moving load and resting on elastic foundation. These analytical and numerical results are of considerable practical importance to apply mathematicians, physicists, transport engineers, construction engineers, designers of structures like railways and highway bridges, roadways and airport runways, to name a few.

3.1 ANALYTICAL RESULT

Equation (24) represents the dynamic response of a simply-supported rectangular plate resting on elastic foundation under the action of a moving concentrated load.

This equation (24) clearly reveals that the amplitude of oscillation of the plate structure is proportional to that of the moving load. Also, the deflection of the plate grows without bounds when

$$\alpha^2 = \gamma_1^2 \text{and} \alpha^2 = \gamma_2^2 \quad (25)$$

wherebya condition known as resonance sets in.

At resonance, one of the natural circular frequencies of the plate structure coincides with the excitation frequency of the moving load, the end result of which is an infinite or excessive deflection of the plate structure.

The relations in equation (25) indicate that the resonant states of the thin plate under the action of moving load are dependent on axial force and foundation stiffness. To this end, one seeks the speeds at which the dynamical system experiences resonance, and so called the critical speed. The critical speeds at the respective resonant states are

$$c_{r_1} = \frac{\alpha-\lambda}{\varphi_j}, \qquad c_{r_2} = \frac{\alpha+\lambda}{\varphi_j} \quad (26)$$

where

$$\varphi_j = \frac{j\pi}{l_x} \quad (27)$$

From equations (26), it is straightforward to examine the influence of the various parameters, such as axial force and foundation stiffness on the critical speeds of the plate dynamical system.

3.2 NUMERICAL ILLUSTRATION

For the purpose of numerical illustration of analytical results obtained for the dynamical plate problem, a uniform plate of length $L_x = 0.459\ m$ and width $L_y = 0.914m$ with bending rigidity $D = 4.943\ \times 10^4 kgm$ is considered. Figure 3.1 indicates the deflection profile of the plate for various values of the axial force N for a fixed value of foundation stiffness. Also, figure 3.2 depicts the deflection profile of the plate for various values of foundation stiffness F for a fixed value of axial force.

From figure 3.1, it is clearly seen that the response amplitude of the plate decreases with increase in the value of axial force for a fixed value of the foundation stiffness. In like manner, the response amplitude decreases with increase in the value of foundation stiffness for a fixed value of axial force.

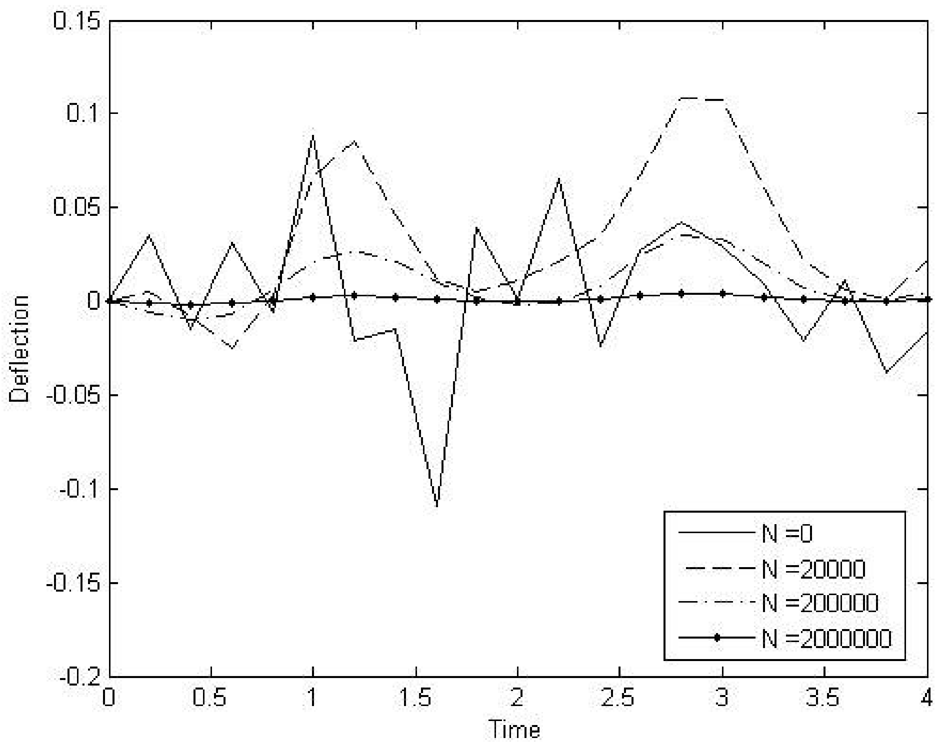


Figure 3.1 Graph showing the deflection profile of simply supported rectangular plate for various values of axial force with fixed value of foundation stiffness.

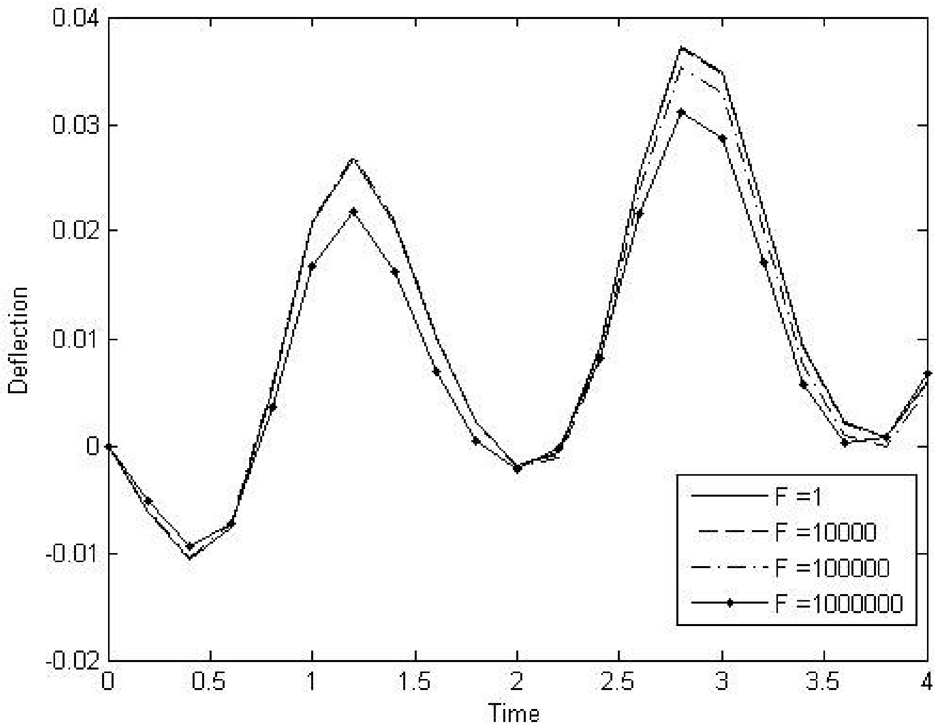


Figure 3.2 Graph showing the deflection profile of simply supported rectangular plate for various values of foundation stiffness with fixed value of axial force.

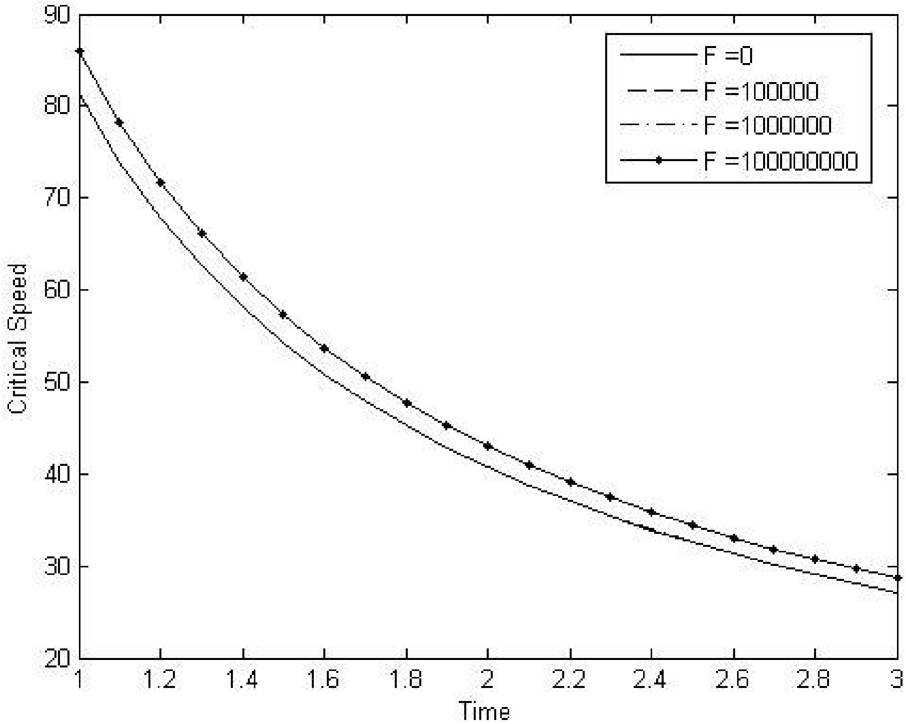


Figure 3.3 Graph showing the profile of the critical speed of the dynamical system for various values of foundation stiffness and fixed value of axial force

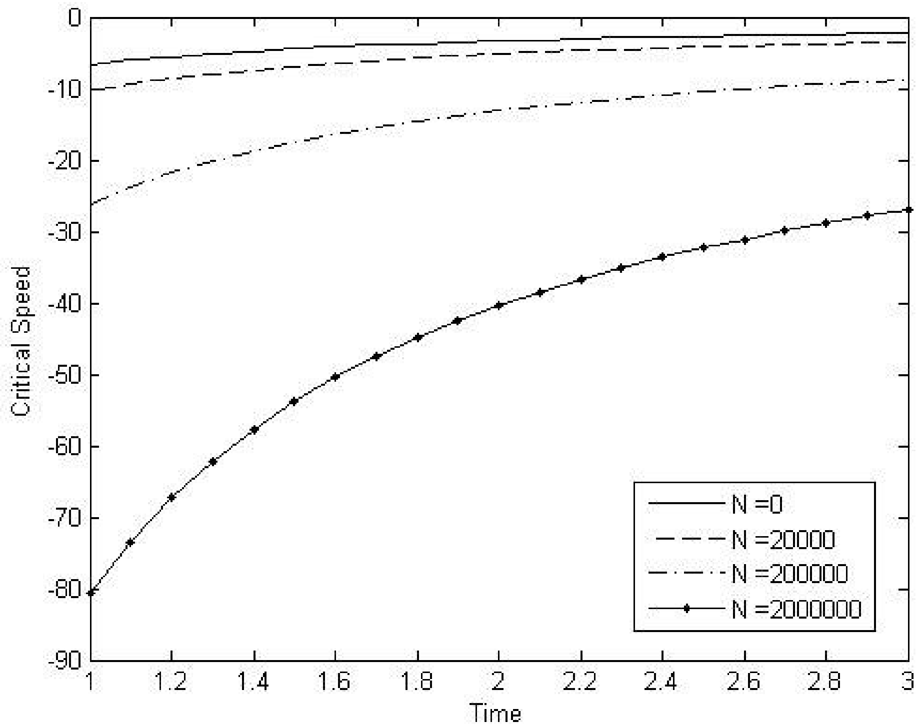


Figure 3.4 Graph showing the profile of the critical speed of the dynamical system for various values of axial force and fixed value of foundation stiffness

4.0 SUMMARY AND CONCLUSION

In this study, the assessment of the response and vibration behaviour of reinforced rectangular plate resting on elastic foundation and traversed by harmonic concentrated load has been presented. Throughout this study, simply-supported rectangular plates are considered. The dynamical plate problem has been solved analytically using the method of integral transformation to produce the expression for the displacement response of the rectangular plate to harmonic concentrated load. From this result, the resonant states and the associated critical speeds were determined and presented graphically. The graphs reveal the effects of the physical phenomena such as axial force and foundation stiffness on the dynamic response of the rectangular plate under the action of harmonic moving loads on one hand and the critical speed of motion on the other. Increase in value of axial force when the value of foundation stiffness is kept constant produced a decrease in the amplitude of deflection of the rectangular plate structure. Thus, prestressed structures experienced reduced deflection andan enhanced load carrying capacity. Also, increase in the value of foundation stiffness when the value of axial force is fixed resulted in decrease in the amplitude of deflection of the plate. Furthermore, the graphs in figures 3.3 and 3.4 revealed that increase in the value of axial force and foundation stiffness produce increase in the critical speed of motion of the dynamic plate indicating the reduction in the attendant risks of resonance.

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